Extensions of modules for $SL(2, K)$

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Proposed running head:

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In this paper, we consider the induced modules ∇ and the Weyl modules Δ for the algebraic group $G = SL(2, K)$ where K is an algebraically closed field of characteristic $p > 0$. We determine the G-modules $H^i(G_1, \nabla(s) \otimes \nabla(t))$ for all $i \geq 0$, where G_1 is the first Frobenius kernel of G. We then use it to find the Ext^1-spaces between twisted tensor products of Weyl modules and induced modules for G. Moreover, we describe explicitly the non-split extensions corresponding to ∇'s.

Key words: special linear group, symmetric powers, decomposition matrix.

Introduction

In the theory of highest weight categories, the classes of modules ∇ and Δ are of central interest. In particular, twisted tensor products of these modules occur as important subquotients of ∇ and Δ (see [12] and [13]).
Here we consider these modules for the group $G = SL(2, K)$,the special linear group of dimension 2 over an algebraically closed field K of characteristic $p > 0$. Suppose that $F : G \longrightarrow G$ is the corresponding Frobenius morphism and let G_1 denote the first Frobenius kernel of G. If V is a G-module then we denote by V^F its Frobenius twist. Considered as a G_1-module, V^F is trivial. Conversely, if W is a G-module on which G_1 acts trivially then $W \cong V^F$ for a unique G-module V and we write $W^{(-1)} := V$.

Consider the Borel subgroup B of G consisting of lower triangular matrices and for $\lambda \in N$, let K_λ denote the 1-dimensional B-module of weight λ. Define the *induced* G-module $\nabla(\lambda)$ by

$$\nabla(\lambda) := \mathrm{Ind}_B^G(K_\lambda).$$

This is isomorphic to the symmetric power $S^\lambda E$ where E is the natural 2-dimensional G-module. The *Weyl* G-modules, $\Delta(\lambda)$, are defined by

$$\Delta(\lambda) := \nabla(\lambda)^*.$$

Note that $\mathrm{soc}\nabla(\lambda) = \mathrm{top}\Delta(\lambda) = L(\lambda)$ is simple and $\{L(\lambda)\,,\, \lambda \in N\}$ form a complete set of non-isomorphic simple G-modules. For $0 \leq \lambda \leq p-1$ we have $L(\lambda) = \nabla(\lambda) = \Delta(\lambda)$ and in general Steinberg's tensor product theorem tells us that if $\lambda = \sum_{i\geq 0} \lambda_i p^i$ is the p-adic expansion of λ then $L(\lambda)$ is given by

$$L(\lambda) = \bigotimes_{i\geq 0} L(\lambda_i)^{F^i}.$$

The simple G-modules are thus self-dual.

The modules $\nabla(\lambda)$ and $\Delta(\lambda)$ have highest weight λ occuring with multiplicity 1 and all their other weights μ satisfy $\mu < \lambda$.

In order to prove our results, we use the Lyndon-Hochschild-Serre 5-term exact sequence relating the Ext^1-spaces of G and G_1. For a rational G-module V, we have the exact sequence (see [3])

$$0 \longrightarrow H^1(G,(V^{G_1})^{(-1)}) \longrightarrow H^1(G,V) \longrightarrow H^1(G_1,V)^G \longrightarrow H^2(G,(V^{G_1})^{(-1)}) \longrightarrow H^2(G,V).$$

In Section 1, we describe properties of G_1-modules and we compute $\mathrm{Ext}^i_{G_1}(\Delta,\nabla)$ for $i \geq 0$ as G-modules. In Section 2, we use the 5-term exact sequence above and the results of Section 1 to compute $\mathrm{Ext}^1_G(\nabla(r)^{F^n} \otimes \Delta(s), \nabla(k)^{F^n} \otimes \nabla(t))$ for $0 \leq k, r$ and $0 \leq s, t \leq p^n - 1$. In particular, we show that it has at most dimension 1. We also find explicitly the non-split extensions corresponding to a ∇. This filtration of ∇ by twisted tensor product of ∇'s and Δ's explains the symmetries observed in the decomposition matrix of G.

1 Computing $\mathrm{Ext}^i_{G_1}(\Delta,\nabla)$

The category of G_1-modules is equivalent to the category of U-modules where U is the restricted Lie algebra of G. In particular, U is a self-injective algebra (see [15]). This category is very well understood ([9],[14]). The simple U-modules are the restriction of the $L(i)$ for $0 \leq i \leq p-1$ and the corresponding projective U-modules $P(i)$ have the following structure: for $0 \leq i \leq p-2$, $\mathrm{soc}P(i) = \mathrm{top}P(i) = L(i)$ and $\mathrm{rad}P(i)/\mathrm{soc}P(i) = L(j) \oplus L(j)$ where $i+j = p-2$ and for $i = p-1$ the projective module $P(p-1) = L(p-1)$ is simple. Thus the projective module $P(p-1)$ is alone in its block and $P(i)$and $P(j)$ belong to the same block if and only if $i = j$ or $i + j = p-2$.

For an indecomposable non-projective U-module M, we denote by $\Omega(M)$ the kernel of the projective cover of M (and we define inductively $\Omega^k(M) = \Omega(\Omega^{k-1}(M))$. Similarly, we define $\Omega^{-1}(M)$ to be the cokernel of the injective hull of M (and we define inductively $\Omega^{-k}(M)$). The projective (injective) G_1-modules are restrictions of G-modules and for $n \geq 0$, we have an exact sequence of G-modules ([17], [4])

$$0 \longrightarrow \nabla(np+i) \longrightarrow P(i) \otimes \nabla(n)^F \longrightarrow \nabla((n+1)p+j) \longrightarrow 0.$$

The restriction of this sequence to G_1 gives the projective cover of $\nabla((n+1)p+j)$ and the injective hull of $\nabla(np+i)$.

The G_1-module $\nabla(np+i)$ has Loewy length 2 for $n \geq 1$. We have a sequence of G-modules ([17], [12])

$$0 \longrightarrow \nabla(n)^F \otimes \nabla(i) \longrightarrow \nabla(np+i) \longrightarrow \nabla(n-1)^F \otimes \Delta(j) \longrightarrow 0 \qquad (1)$$

and its restriction to G_1 gives the Loewy series of $\nabla(np+i)$ as a G_1-module.

Note finally that if V, W and X are G-modules and $n \geq 0$ then $\mathrm{Ext}^n_{G_1}(V, W)$

has a natural structure of G-module and

$$\mathrm{Ext}^n_{G_1}(V, W \otimes X^F) \cong \mathrm{Ext}^n_{G_1}(V, W) \otimes X^F$$

as G-modules.

W. van der Kallen proved in [16] that if V is a G-module with a good filtration (that is a filtration with quotients isomorphic to some ∇'s) then $H^0(G_1, V)^{(-1)}$ has a good filtration and hence, by dimension shifting (see [7]), $H^i(G_1, V)^{(-1)}$ has a good filtration for all $i \geq 0$. Note that the module $V = \nabla \otimes \nabla$ has a good filtration and the next two Propositions give the G-modules $H^i(G_1, V) = \mathrm{Ext}^i_{G_1}(\Delta, \nabla)$ for $i \geq 0$.

Write $t = t_1 p + t_0$ and $s = s_1 p + s_0$ where $0 \leq s_0, t_0 \leq p - 1$.

Proposition 1.1 *For $i \geq 1$ we have*

$$\mathrm{Ext}^i_{G_1}(\Delta(s), \nabla(t)) \cong \begin{cases} \nabla(s_1 + t_1 + i)^F & \textit{if } s_0 + t_0 = p - 2 \textit{ and } i \textit{ odd} \\ & \textit{or } s_0 = t_0 \leq p - 2 \textit{ and } i \textit{ even} \\ 0 & \textit{otherwise.} \end{cases}$$

Proof:

From the block structure of G_1 we only need to consider the cases $s_0 = t_0$ and $s_0 + t_0 = p - 2$. Note that if $s_0 = t_0 = p - 1$ then $\Delta(s)$ and $\nabla(t)$ are projective and so there is no non-split extension. Now suppose $s_0, t_0 \leq p - 2$.

$$\mathrm{Ext}^i_{G_1}(\Delta(s_1 p + s_0), \nabla(t_1 p + t_0) \cong \mathrm{Ext}^i_{G_1}(\Omega^{-s_1}(\Delta(s_1 p + s_0), \Omega^{-s_1}(\nabla(t_1 p + t_0)))$$

$$\cong \begin{cases} \mathrm{Ext}^i_{G_1}(\Delta(s_0), \nabla((s_1+t_1)p+t_0)) & \text{if } s_1 \text{ even} \\ \mathrm{Ext}^i_{G_1}(\Delta(p-2-s_0), \nabla((s_1+t_1)p+p-2-t_0)) & \text{if } s_1 \text{ odd} \end{cases}$$

Now consider the exact sequence,

$$0 \to \nabla((s_1+t_1)p+t_0) \to P(t_0)\otimes\nabla(s_1+t_1)^F \to \nabla((s_1+t_1+1)p+p-2-t_0) \to 0$$

and apply $\mathrm{Hom}_{G_1}(\Delta(s_0), -)$ to get

$$0 \to \mathrm{Hom}_{G_1}(\Delta(s_0), \nabla((s_1+t_1)p+t_0)) \to \mathrm{Hom}_{G_1}(\Delta(s_0), P(t_0)\otimes\nabla(s_1+t_1)^F)$$

$$\longrightarrow \mathrm{Hom}_{G_1}(\Delta(s_0), \nabla((s_1+t_1+1)p+p-2-t_0))$$

$$\longrightarrow \mathrm{Ext}^1_{G_1}(\Delta(s_0), \nabla((s_1+t_1)p+t_0)) \to 0 \tag{2}$$

and

$$\mathrm{Ext}^{i+1}_{G_1}(\Delta(s_0), \nabla((s_1+t_1)p+t_0)) \cong \mathrm{Ext}^i_{G_1}(\Delta(s_0), \nabla((s_1+t_1+1)p+p-2-t_0)).$$

Thus, if we prove the case $i = 1$ then the result follows by induction. Now, observe that in the exact sequence (2) the first two terms are isomorphic ($\Delta(s_0)$ is simple and $P(t_0)\otimes\nabla(s_1+t_1)^F$ is the injective hull of $\nabla((s_1+t_1)p+t_0)$), hence the last two terms are isomorphic too and we get

$$\begin{aligned} &\mathrm{Ext}^1_{G_1}(\Delta(s_0), \nabla((s_1+t_1)p+t_0)) \\ &\cong \mathrm{Hom}_{G_1}(\Delta(s_0), \nabla((s_1+t_1+1)p+p-2-t_0)) \\ &\cong \mathrm{Hom}_{G_1}(\Delta(s_0), P(p-2-t_0)\otimes\nabla(s_1+t_1+1)^F) \\ &\cong \mathrm{Hom}_{G_1}(\Delta(s_0), P(p-2-t_0))\otimes\nabla(s_1+t_1+1)^F \\ &\cong \begin{cases} \nabla(s_1+t_1+1)^F & \text{if } s_0+t_0 = p-2 \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

The proposition then follows by induction on i. QED

Proposition 1.2

$$\mathrm{Hom}_{G_1}(\Delta(s_1p+s_0),\nabla(t_1p+t_0)) \cong \begin{cases} (\nabla(s_1)\otimes\nabla(t_1))^F & \text{if } s_0=t_0 \\ 0 & \text{otherwise.} \end{cases}$$

Proof:

Note that by the decomposition into blocks of G_1, we only need to consider the cases $s_0+t_0=p-2$ and $s_0=t_0$. Suppose for a start that $s_0,t_0\le p-2$. Consider the exact sequence

$$0\longrightarrow \nabla(t_1)^F\otimes\nabla(t_0)\longrightarrow\nabla(t_1p+t_0)\longrightarrow\nabla(t_1-1)^F\otimes\Delta(p-2-t_0)\longrightarrow 0.$$

Apply $\mathrm{Hom}_{G_1}(\Delta(s_1p+s_0),-)$ to get the exact sequence

$$0\to\mathrm{Hom}_{G_1}(\Delta(s_1p+s_0),\nabla(t_1)^F\otimes\nabla(t_0))\to\mathrm{Hom}_{G_1}(\Delta(s_1p+s_0),\nabla(t_1p+t_0))$$

$$\to\mathrm{Hom}_{G_1}(\Delta(s_1p+s_0),\nabla(t_1-1)^F\otimes\Delta(p-2-t_0))\to\mathrm{Ext}^1_{G_1}(\Delta(s_1p+s_0),\nabla(t_1)^F\otimes\nabla(t_0))$$

$$\to\mathrm{Ext}^1_{G_1}(\Delta(s_1p+s_0),\nabla(t_1p+t_0)). \tag{3}$$

Now,

$$\mathrm{Hom}_{G_1}(\Delta(s_1p+s_0),\nabla(t_1)^F\otimes\nabla(t_0))\cong\mathrm{Hom}_{G_1}(\nabla(t_0),\nabla(s_1p+s_0))\otimes\nabla(t_1)^F$$

$$\cong\mathrm{Hom}_{G_1}(\nabla(t_0),P(s_0))\otimes\nabla(s_1)^F\otimes\nabla(t_1)^F$$

$$\cong\begin{cases} (\nabla(s_1)\otimes\nabla(t_1))^F & \text{if } s_0=t_0 \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\begin{aligned}
&\mathrm{Hom}_{G_1}(\Delta(s_1p+s_0),\nabla(t_1-1)^F\otimes\Delta(p-2-t_0))\\
&\cong \mathrm{Hom}_{G_1}(\nabla(p-2-t_0),\nabla(s_1p+s_0))\otimes\nabla(t_1-1)^F\\
&\cong \mathrm{Hom}_{G_1}(\nabla(p-2-t_0),P(s_0))\otimes\nabla(s_1)^F\otimes\nabla(t_1-1)^F\\
&\cong \begin{cases} (\nabla(s_1)\otimes\nabla(t_1-1))^F & \text{if } s_0+t_0=p-2\\ 0 & \text{otherwise.}\end{cases}
\end{aligned}$$

Using Proposition 1.1, we get

$$\begin{aligned}
\mathrm{Ext}^1_{G_1}(\Delta(s_1p+s_0),\nabla(t_1)^F\otimes\nabla(t_0)) &\cong \mathrm{Ext}^1_{G_1}(\nabla(t_0),\nabla(s_1p+s_0))\otimes\nabla(t_1)^F\\
&\cong \begin{cases} (\nabla(s_1+1)\otimes\nabla(t_1))^F & \text{if } s_0+t_0=p-2\\ 0 & \text{otherwise}\end{cases}
\end{aligned}$$

and

$$\mathrm{Ext}^1_{G_1}(\Delta(s_1p+s_0),\nabla(t_1p+t_0))\cong \begin{cases} \nabla(s_1+t_1+1)^F & \text{if } s_0+t_0=p-2\\ 0 & \text{otherwise.}\end{cases}$$

So if $s_0+t_0=p-2$ and $p>2$ (i.e. $s_0\neq t_0$), then the exact sequence (3) becomes

$$\begin{aligned}
0 \longrightarrow\ & \mathrm{Hom}_{G_1}(\Delta(s_1p+s_0),\nabla(t_1p+t_0))\longrightarrow(\nabla(s_1)\otimes\nabla(t_1-1))^F\\
\longrightarrow\ & (\nabla(s_1+1)\otimes\nabla(t_1))^F\longrightarrow\nabla(s_1+t_1+1)^F.
\end{aligned}$$

As

$$\dim(\nabla(s_1+1)\otimes\nabla(t_1))^F=\dim(\nabla(s_1)\otimes\nabla(t_1-1))^F+\dim\nabla(s_1+t_1+1)^F,$$

we deduce that

$$\mathrm{Hom}_{G_1}(\Delta(s_1p+s_0),\nabla(t_1p+t_0))=0.$$

If $s_0 = t_0$ and $p = 2$, the exact sequence (3) has the form

$$\begin{aligned} 0 \quad &\longrightarrow \quad (\nabla(s_1) \otimes \nabla(t_1))^F \longrightarrow \mathrm{Hom}_{G_1}(\Delta(s_1 2 + s_0), \nabla(t_1 2 + t_0)) \\ &\longrightarrow \quad (\nabla(s_1) \otimes \nabla(t_1 - 1))^F \longrightarrow (\nabla(s_1 + 1) \otimes \nabla(t_1))^F \longrightarrow \nabla(s_1 + t_1 + 1)^F. \end{aligned}$$

Hence,

$$\mathrm{Hom}_{G_1}(\Delta(s_1 2 + s_0), \nabla(t_1 2 + t_0)) \cong (\nabla(s_1) \otimes \nabla(t_1))^F.$$

Finally if $s_0 = t_0$ and $p > 2$ then clearly

$$\mathrm{Hom}_{G_1}(\Delta(s_1 p + s_0), \nabla(t_1 p + t_0)) \cong (\nabla(s_1) \otimes \nabla(t_1))^F.$$

In the case where $s_0 = t_0 = p - 1$, we have the following

$$\begin{aligned} \Delta(s_1 p + s_0) \quad &\cong \quad \Delta(s_1)^F \otimes \Delta(p - 1) \\ \nabla(t_1 p + t_0) \quad &\cong \quad \nabla(t_1)^F \otimes \nabla(p - 1), \end{aligned}$$

and so

$$\begin{aligned} &\mathrm{Hom}_{G_1}(\Delta(s_1 p + (p - 1)), \nabla(t_1 p + (p - 1))) \\ &\cong \mathrm{Hom}_{G_1}(\Delta(p - 1), \nabla(p - 1)) \otimes (\nabla(s_1) \otimes \nabla(t_1))^F \\ &\cong (\nabla(s_1) \otimes \nabla(t_1))^F. \end{aligned}$$

This completes the proof. QED

2 Extensions of G-modules

In [5] and [8], Cox and Erdmann determined the Ext^1 and the Hom spaces between $\nabla(\lambda)$ and $\nabla(\mu)$ for arbitrary weights λ and μ. For completeness and to fix our notation, we state their result here.

For $0 \leq a \leq p-1$ denote by $\hat{a}$, the integer such that $a + \hat{a} = p-1$. For a weight μ, define

$$\psi^0(\mu) = \left\{ \sum_{i=0}^{u-1} \hat{\mu_i} p^i \ : \ u \geq 0 \right\}$$

and

$$\psi^1(\mu) = \left\{ \sum_{i=0}^{u-1} \hat{\mu_i} p^i + p^{u+a} \ : \ \hat{\mu_u} \neq 0, \ , a \geq 1, \ u \geq 0 \right\} \cup \left\{ \sum_{i=0}^{u} \hat{\mu_i} p^i \ : \ \hat{\mu_u} \neq 0, \ u \geq 0 \right\}.$$

With this notation we have,

$$\mathrm{Hom}_G(\nabla(\lambda), \nabla(\mu)) \quad \cong \quad \begin{cases} K & \text{if} \quad \lambda = \mu + 2d, \ d \in \psi^0(\mu) \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

and

$$\mathrm{Ext}^1_G(\nabla(\lambda), \nabla(\mu)) \quad \cong \quad \begin{cases} K & \text{if} \quad \lambda = \mu + 2e, \ e \in \psi^1(\mu) \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

In [2], Cline determined all the Ext^1-spaces between simple G-modules. In particular, for simple modules $\nabla(r)^F \otimes \nabla(s)$ and $\nabla(k)^F \otimes \nabla(t)$, he proved that

$$\mathrm{Ext}^1_G(\nabla(r)^F \otimes \nabla(s), \nabla(k)^F \otimes \nabla(t)) \cong \begin{cases} K & \text{if } r = k \pm 1, \ s + t = p - 2 \\ 0 & \text{otherwise} \end{cases}$$

The following theorem extends this result.

Theorem 2.1 *Let $0 \leq k, r$ and $0 \leq s, t \leq p^n - 1$ then we have*

$$\mathrm{Ext}^1_G(\nabla(r)^{F^n} \otimes \Delta(s), \nabla(k)^{F^n} \otimes \nabla(t)) \cong \begin{cases} K \quad \text{if} \quad \begin{array}{l} r = k + 2e, \ e \in \psi^1(k) \\ s = t \end{array} \\ \\ \quad\ \ \text{or} \quad \begin{array}{l} r = k \pm 1 + 2d, \ d \in \psi^0(k) \\ t = t_0 + t_1 p^i, \ 0 \leq t_0 \leq p^i - 1 \\ s = t_0 + (p^{n-i} - 2 - t_1)p^i \end{array} \\ \\ 0 \quad \text{otherwise} \end{cases}$$

Proof:

In order to prove this theorem, we use the five terms exact sequence:

$$0 \longrightarrow H^1(G, (V^{G_1})^{(-1)}) \longrightarrow H^1(G, V) \longrightarrow H^1(G_1, V)^G \longrightarrow H^2(G, (V^{G_1})^{(-1)}) \longrightarrow H^2(G, V),$$

with $V = \Delta(r)^{F^n} \otimes \nabla(k)^{F^n} \otimes \nabla(s) \otimes \nabla(t)$.

Write $s = s_1 p + s_0$ and $t = t_1 p + t_0$. Let us first compute $H^1(G, (V^{G_1})^{(-1)})$.

Using Proposition 1.2, we have

$$\begin{aligned} V^{G_1} &= \mathrm{Hom}_{G_1}(\Delta(s), \nabla(t)) \otimes \Delta(r)^{F^n} \otimes \nabla(k)^{F^n} \\ &\cong \begin{cases} (\nabla(s_1) \otimes \nabla(t_1))^F \otimes \Delta(r)^{F^n} \otimes \nabla(k)^{F^n} & \text{if } s_0 = t_0 \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Now,

$$(V^{G_1})^{(-1)} \cong \begin{cases} \nabla(s_1) \otimes \nabla(t_1) \otimes \Delta(r)^{F^{n-1}} \otimes \nabla(k)^{F^{n-1}} & \text{if } s_0 = t_0 \\ 0 & \text{otherwise.} \end{cases}$$

Hence for $s_0 = t_0$ we have

$$H^1(G, (V^{G_1})^{(-1)}) \cong \mathrm{Ext}^1_G(\nabla(r)^{F^{n-1}} \otimes \Delta(s_1), \nabla(k)^{F^{n-1}} \otimes \nabla(t_1)),$$

and is zero in all other cases.

Let us now compute $H^1(G_1, V)^G$. Using Proposition 1.1, we have

$$\begin{aligned} H^1(G_1, V) &= \mathrm{Ext}^1_{G_1}(\nabla(r)^{F^n} \otimes \Delta(s), \nabla(k)^{F^n} \otimes \nabla(t)) \\ &\cong \mathrm{Ext}^1_{G_1}(\Delta(s), \nabla(t)) \otimes \Delta(r)^{F^n} \otimes \nabla(k)^{F^n} \\ &\cong \begin{cases} \nabla(s_1 + t_1 + 1)^F \otimes \Delta(r)^{F^n} \otimes \nabla(k)^{F^n} & \text{if } s_0 + t_0 = p - 2 \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Thus,

$$H^1(G_1, V)^G \cong \begin{cases} \mathrm{Hom}_G(\Delta(s_1 + t_1 + 1)^F, \Delta(r)^{F^n} \otimes \nabla(k)^{F^n}) & \text{if } s_0 + t_0 = p - 2 \\ 0 & \text{otherwise} \end{cases}$$

Note that all the weights of $\Delta(r)^{F^n} \otimes \nabla(k)^{F^n}$ are multiples of p^n, so to get non-zero homomorphisms, we must have $s_1 + t_1 + 1 = cp^{n-1}$ for some c. But $s, t \leq p^n - 1$ implies that $s_1 + t_1 \leq 2p^{n-1} - 2$, thus $c = 1$ and $s_1 + t_1 + 1 = p^{n-1}$. Observe that

$$\mathrm{Hom}_G(\Delta(p^{n-1})^F, \Delta(r)^{F^n} \otimes \nabla(k)^{F^n}) \cong \mathrm{Hom}_G(\nabla(r)^{F^n}, \nabla(p^{n-1})^F \otimes \nabla(k)^{F^n})$$

and that all the weights of $\nabla(r)^{F^n}$ are multiple of p^n so the image of a homomorphism from $\nabla(r)^{F^n}$ to $\nabla(p^{n-1})^F \otimes \nabla(k)^{F^n}$ lies in the submodule $\nabla(1)^{F^n} \otimes \nabla(k)^{F^n} \leq \nabla(p^{n-1})^F \otimes \nabla(k)^{F^n}$. Hence,

$$\begin{aligned} \mathrm{Hom}_G(\Delta(p^{n-1})^F, \Delta(r)^{F^n} \otimes \nabla(k)^{F^n}) &\cong \mathrm{Hom}_G(\nabla(r)^{F^n}, \nabla(1)^{F^n} \otimes \nabla(k)^{F^n}) \\ &\cong \mathrm{Hom}_G(\nabla(r), \nabla(1) \otimes \nabla(k)). \end{aligned}$$

We claim that $\mathrm{Hom}_G(\nabla(r), \nabla(1) \otimes \nabla(k)) \cong K$ if $r = k \pm 1 + 2d$ where $d \in \psi^0(k)$ and zero otherwise. Consider the exact sequence

$$0 \longrightarrow \nabla(z-1) \longrightarrow \nabla(1) \otimes \nabla(z) \longrightarrow \nabla(z+1) \longrightarrow 0. \tag{6}$$

This sequence splits if and only if $z \neq -1(\mathrm{mod}\ p)$. Note that for $\mathrm{Hom}_G(\nabla(r), \nabla(1) \otimes \nabla(k))$ to be non zero, we must have $r + k = 1(\mathrm{mod}\ 2)$. Now suppose $k = -1(\mathrm{mod}\ p)$ then we can assume $r \neq -1(\mathrm{mod}\ p)$ and so using (6) with $z = r$ we have

$$\begin{aligned} \mathrm{Hom}_G(\nabla(r), \nabla(1) \otimes \nabla(k)) &\cong \mathrm{Hom}_G(\nabla(1) \otimes \nabla(r), \nabla(k)) \\ &\cong \mathrm{Hom}_G(\nabla(r-1) \oplus \nabla(r+1), \nabla(k)). \end{aligned}$$

Now, using (4) we deduce that $\mathrm{Hom}_G(\nabla(r-1), \nabla(k)) \cong K$ if and only if $r - 1 = k + 2d$ where $d \in \psi^0(k)$ and it is zero otherwise, and $\mathrm{Hom}_G(\nabla(r+1), \nabla(k)) \cong K$ if and only if $r + 1 = k + 2d'$ where $d' \in \psi^0(k)$ and zero otherwise. Suppose they are both non-zero then $k + 1 + 2d = k - 1 + 2d'$. But this can only happen when $d = 0$, $d' = 1$ and $r = k + 1$. This means that $k = p - 2(\mathrm{mod}\ p)$ and $r = -1(\mathrm{mod}\ p)$ contradicting our assumption. Now if $k \neq -1(\mathrm{mod}\ p)$ we use (6) with $z = k$ and the claim follows by a similar argument.

Hence, we have proved the following

$$H^1(G_1, V)^G \cong \begin{cases} K & \text{if } s_0 + t_0 = p - 2,\ s_1 + t_1 = p^{n-1} - 1 \\ & r = k \pm 1 + 2d \text{ where } d \in \psi^0(k) \\ & \\ 0 & \text{otherwise.} \end{cases}$$

Let us now use the five term sequence to determine $H^1(G, V)$. We shall do this by induction on n. For $n = 1$ we have $s, t \leq p - 1$ and

$$H^1(G, (V^{G_1})^{(-1)}) \cong \begin{cases} K & \text{if } r = k + 2e,\ e \in \psi^1(k) \text{ and } s = t \\ 0 & \text{otherwise} \end{cases}$$

and

$$H^1(G_1, V)^G \cong \begin{cases} K & \text{if } r = k \pm 1 + 2d,\ d \in \psi^0(k) \text{ and } s + t = p - 2 \\ 0 & \text{otherwise,} \end{cases}$$

thus,

$$H^1(G, V) \cong \begin{cases} K & \text{if } r = k + 2e,\ e \in \psi^1(k) \text{ and } s = t \\ & \text{or } r = k \pm 1 + 2d,\ d \in \psi^0(k) \text{ and } s + t = p - 2 \\ 0 & \text{otherwise.} \end{cases}$$

Now we use induction. Note that if $p = 2$ and $s_0 = t_0 = 0$ and $s_1 + t_1 = 2^{n-1} - 1$ then $\Delta(s_1)$ and $\nabla(t_1)$ are in different blocks of G_1 and so

$$\mathrm{Ext}^i_G(\nabla(r)^{F^{n-1}} \otimes \Delta(s_1), \nabla(k)^{F^{n-1}} \otimes \nabla(t_1)) = 0 \quad \text{for all } i.$$

So for all prime p we get

$$H^1(G,V) \cong \begin{cases} K \quad \text{if} \quad \begin{matrix} r = k+2e, \ e \in \psi^1(k) \\ s = t \end{matrix} \\ \\ \qquad \text{or} \quad \begin{matrix} r = k \pm 1 + 2d, \ d \in \psi^0(k) \\ t = t_0 + t_1 p^i, \ 0 \le t_0 \le p^i - 1 \\ s = t_0 + (p^{n-i} - 2 - t_1)p^i \end{matrix} \\ \\ 0 \quad \text{otherwise.} \end{cases}$$

This completes the proof of our theorem. QED

Note that if we set $n = 0$ and $s = t = 0$ in Theorem 2.1 we get Erdmann and Cox's result given by equation (5).

The following proposition shows that when $r = k - 1$ and $s = p^n - 2 - t$, the extension is given by $\nabla(kp^n + t)$. By considering weights, it is easy to see that no other extension described in Theorem 2.1 can be isomorphic to an induced module $\nabla(\lambda)$.

Proposition 2.1 *For $k \in N$ and $0 \le t \le p^n - 2$, there is an exact sequence of G-modules*

$$0 \longrightarrow \nabla(k)^{F^n} \otimes \nabla(t) \longrightarrow \nabla(kp^n + t) \longrightarrow \nabla(k-1)^{F^n} \otimes \Delta(p^n - 2 - t) \longrightarrow 0.$$

Moreover, $\nabla(kp^n + t)$ is the only non-split extension, up to isomorphism, of $\nabla(k-1)^{F^n} \otimes \Delta(p^n - t - 2)$ by $\nabla(k)^{F^n} \otimes \nabla(t)$.

Dually, the only non-split extension, up to isomorphism, of $\Delta(k)^{F^n} \otimes \Delta(t)$ *by* $\Delta(k-1)^{F^n} \otimes \nabla(p^n - t - 2)$ *is given by* $\Delta(kp^n + t)$.

Remark 1: For $k \in N$ we have an isomorphism between $\nabla(k-1)^{F^n} \otimes St_n$ and $\nabla(kp^n - 1)$ given by multiplication of polynomials. It is known that there is an isomorphism between these modules more generally, see for example [11](II.3).

Proof of Proposition 2.1:

If $n = 1$ then we are done by (1) (Section 1). Suppose $n > 1$ and write $t = ap^{n-1} + d$, for $0 \le a \le p-1$ and $0 \le d \le p^{n-1} - 1$. Using induction we have an exact sequence

$$0 \longrightarrow \nabla(kp+a)^{F^{n-1}} \otimes \nabla(d) \longrightarrow \nabla((kp+a)p^{n-1} + d) \longrightarrow$$
$$\nabla(kp + (a-1))^{F^{n-1}} \otimes \Delta(p^{n-1} - d - 2) \longrightarrow 0.$$

Using the exact sequences (1) for $\nabla(kp+a)^{F^{n-1}}$ and $\nabla(kp+(a-1))^{F^{n-1}}$ we get a filtration of $\nabla(kp^n + ap^{n-1} + d)$ with quotients

$$\nabla(k-1)^{F^n} \otimes \Delta(p-a-1)^{F^{n-1}} \otimes \Delta(p^{n-1} - d - 2)$$
$$\nabla(k)^{F^n} \otimes \nabla(a-1)^{F^{n-1}} \otimes \Delta(p^{n-1} - d - 2)$$
$$\nabla(k-1)^{F^n} \otimes \Delta(p-a-2)^{F^{n-1}} \otimes \nabla(d)$$
$$\nabla(k)^{F^n} \otimes \nabla(a)^{F^{n-1}} \otimes \nabla(d)$$

Observe that the module $\nabla(kp^n + ap^{n-1} + d)$ is multiplicity-free, so that the four quotients have disjoint sets of weights. Hence, $\nabla(kp^n + t)/\nabla(k)^{F^n} \otimes \nabla(t)$

has a filtration with quotients

$$\nabla(k-1)^{F^n} \otimes \Delta(p-a-1)^{F^{n-1}} \otimes \Delta(p^{n-1}-d-2)$$
$$\nabla(k-1)^{F^n} \otimes \Delta(p-a-2)^{F^{n-1}} \otimes \nabla(d)$$

Note that for $a = p-1$ or $d = p^{n-1}-1$, we only have one factor appearing and so we are done by Remark 1 above. So suppose $a \leq p-2$ and $d \leq p^{n-1}-2$. Using a very similar argument to the proof of Theorem 2.1 we can show that

$$\begin{aligned}\mathrm{Ext}^1_G(\nabla(k-1)^{F^n} \otimes \Delta(p-a-1)^{F^{n-1}} \otimes \Delta(p^{n-1}-d-2),& \\ \nabla(k-1)^{F^n} \otimes \Delta(p-a-2)^{F^{n-1}} \otimes \nabla(d)) \qquad &\cong K.\end{aligned}$$

Now as $\nabla(kp^n+t)$ has simple top (see [1]), $\nabla(kp^n+t)/\nabla(k)^{F^n} \otimes \nabla(t)$ cannot be a direct sum of non-zero modules. By induction, we know that $\Delta(p^n - ap^{n-1} - d - 2)$ has a filtration with quotients

$$\Delta(p-a-1)^{F^{n-1}} \otimes \Delta(p^{n-1}-d-2)$$
$$\Delta(p-a-2)^{F^{n-1}} \otimes \nabla(d)$$

We deduce that the quotient $\nabla(kp^n+t)/\nabla(k)^{F^n} \otimes \nabla(t)$ is isomorphic to

$$\nabla(k-1)^{F^n} \otimes \Delta(p^n - ap^{n-1} - d - 2) = \nabla(k-1)^{F^n} \otimes \Delta(p^n - 2 - t).$$

This completes the proof QED

Remark 2: S.Donkin suggested an alternative proof of Proposition 2.1. I shall sketch his argument here. Let us start with the exact sequence of B-modules

$$0 \longrightarrow \nabla(s-1) \otimes K_{-1} \longrightarrow \nabla(s) \longrightarrow K_s \longrightarrow 0 \qquad (7)$$

for any positive integer s. Apply the Frobenius morphism F^n to the sequence (7) and tensor it with K_r for some $0 \leq r \leq p^n - 1$. Then applying the induction functor from B-modules to G-modules and using the duality of induction (see [11],II.4)gives the required sequence.

Remark 3: The composition factors of the ∇'s are known for $SL(2, K)$ (use for example equation (1) repeatedly) but Proposition 2.1 gives a direct explanation of the symmetries observed by A.Henke in the decomposition matrix of SL(2,K) (see [10]). More precisely, if we write $\lambda = kp^n + t$ with $k \leq p - 1$ then our proposition tells us that

$$[\nabla(kp^n + t) : L(kp^n + a)] = [\nabla(t) : L(a)],$$

$$[\nabla(kp^n + t) : L((k-1)p^n + b)] = [\nabla(p^n - 2 - t) : L(b)].$$

Let us write the decomposition matrix of G with the ∇'s on the horizontal axis and the L's on the vertical axis (see figures 1 and 2 below). Then for each $n \geq 1$ and each $1 \leq k \leq p - 1$, the columns corresponding to $\nabla(kp^n + t)$ for $0 \leq t \leq p^n - 1$ are obtained from the left bottom $p^n \times p^n$ block by

1. Translation of length k along the diagonal,
2. Translation of length $k - 1$ along the diagonal and then reflection through the column corresponding to $\nabla(kp^n - 1)$ (shaded on the figures).

Hence, we can construct the decomposition matrix inductively starting with the left bottom $p \times p$ block which is just a diagonal matrix, as for $0 \leq r \leq p - 1$ we have $\nabla(r) = L(r)$.

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